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Gauge Transformations in Quantum Mechanics and the Unification of Nonlinear Schrödinger Equations

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Abstract

Beginning with ordinary quantum mechanics for spinless particles, together with the hypothesis that all experimental measurements consist of positional measurements at different times, we characterize directly a class of nonlinear quantum theories physically equivalent to linear quantum mechanics through nonlinear gauge transformations. We show that under two physically-motivated assumptions, these transformations are uniquely determined: they are exactly the group of time-dependent, nonlinear gauge transformations introduced previously for a family of nonlinear Schrödinger equations. The general equation in this family, including terms considered by Kostin, by Bialynicki-Birula and Mycielski, and by Doebner and Goldin, with time-dependent coefficients, can be obtained from the linear Schrödinger equation through gauge transformation and a subsequent process we call *gauge generalization.* We thus unify, on fundamental grounds, a rather diverse set of nonlinear time-evolutions in quantum mechanics.

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I. INTRODUCTION

Recently a group $\mathcal{N}$ of nonlinear gauge transformations was introduced and shown to act as a transformation group in a family $\mathcal{F}$ of nonlinear Schrödinger equations (NLSEs). [1] The family $\mathcal{F}$ consists of equations with nonlinear terms of the type introduced by Kostin [2], by Bialynicki-Birula and Mycielski [3], by Guerra and Pusterla [4], and by Doebner and Goldin [5,6], with time-dependent coefficients.

A transformation $N_{(\gamma,\Lambda)} \in \mathcal{N}$ is labeled by two real, time-dependent parameters γ and Λ (with $\Lambda \neq 0$), and acts as a nonlinear analogue of a gauge transformation in quantum mechanics. Letting the time-dependent wave function $\psi(\mathbf{x},t)$ on $\mathbf{R}^3$ be an arbitrary solution of any particular NLSE in $\mathcal{F}$, $N_{(\gamma,\Lambda)}$ is given by

$$\psi' = N_{(\gamma,\Lambda)}[\psi] = |\psi| \, \exp\left[\, i(\gamma \ln|\psi| + \Lambda \arg\psi)\,\right]. \tag{1}$$

Then ψ' solves a transformed equation that also belongs to $\mathcal{F}$.

The physical interpretation of this construction, developed briefly below, was elaborated in some detail in Ref. [1]. However, the underlying mathematical structure, and the physical reasons for the form of (1), remained somewhat hidden. Eq. (1) was motivated in earlier work by the desire to linearize the equations in a special subset of $\mathcal{F}$, and to obtain stationary and nonstationary solutions [6–11]. The present paper takes a different, more fundamental approach to nonlinear gauge transformations and their consequences.

We begin with linear, nonrelativistic quantum mechanics for spinless particles in $\mathbf{R}^3$, together with the assumption, advocated for instance in Refs. [12–14], and discussed in Refs. [1,9,10,15,16], that all experimental measurements consist fundamentally of positional measurements made at different times. Defining as usual the positional probability density $\rho\,(\mathbf{x},t) = \overline{\psi(\mathbf{x},t)}\psi(\mathbf{x},t)$, where ψ conventionally is a normalized solution of the linear Schrödinger equation, we are therefore interested in transformations N which leave $\rho\,(\mathbf{x},t)$ invariant — i.e., such that for all ψ in an appropriate domain of the unit sphere in the Hilbert space $\mathcal{H}$,

$$\overline{N[\psi](\mathbf{x},t)}N[\psi](\mathbf{x},t) = \overline{\psi(\mathbf{x},t)}\psi(\mathbf{x},t)\,. \tag{2}$$

In addition, N should respect the prescription for writing the wave function subsequent to an ideal positional measurement. A conventional prescription for such a measurement at time t_1 consists of a projection in a region B of position space (a Borel subset of $\mathbf{R}^3$), with normalization,

$$\psi_s(\mathbf{x},t_1) = \begin{cases} \dfrac{\psi(\mathbf{x},t_1)}{\left(\int_B |\psi(\mathbf{x},t_1)|^2 d^3x\right)^{1/2}}\,, & \mathbf{x} \in B \\[2ex] 0\,, & \mathbf{x} \notin B \end{cases} \tag{3}$$

followed by time-evolution of $\psi_s(\mathbf{x},t)$ for $t > t_1$ (here the subscript "s" stands for "subsequent"). As N should respect this prescription, we need for all ψ, $\mathbf{x}$ and $t \geq t_1$,

$$|N[\psi]_s(\mathbf{x},t)|^2 = |\psi_s(\mathbf{x},t)|^2\,, \tag{4}$$

and because of (2),

$$|N[\psi]_s(\mathbf{x},t)|^2 = |N[\psi_s](\mathbf{x},t)|^2 . \tag{5}$$

We remark that in writing (3) we do not intend to express a commitment to a particular formalism for describing measurement. We merely note that the justification of N as as a gauge transformation requires that in addition to (2) it leave invariant the outcomes of *sequences* of positional measurements at various times. Eq. (3) is one prescription for predicting such outcomes in quantum mechanics.

Now if all actual measurements (outcomes of experiments) are obtained from positional measurements performed at various times, it can be argued that a system with states ψ obeying the Schrödinger equation, and one with states $N[\psi]$ obeying a transformed equation, have the same physical content. But we make two essential observations:

(a) Eqs. (2) and (4)-(5) do not require N to be a linear transformation — nonlinear N are also possible.

(b) Such nonlinear choices of N will transform a system governed by the usual, linear Schrödinger equation to physically equivalent systems obeying NLSEs that are, of course, linearizable (by construction).

The usual formulation and interpretation of quantum mechanics is based quite deeply on linearity and linear structures — superposition principle, on observables modeled by self-adjoint linear operators, on a linear time-evolution equation for the states, on a measurement process involving orthogonal projection onto linear subspaces for all sorts of observables, and on the description of mixed states by density matrices. Any proposal for nonlinearity in quantum mechanics requires a revised mathematical formulation and physical interpretation of all these ideas. Here the linearizable NLSEs obtained using N can be useful. Due to their physical equivalence with linear quantum mechanics, they serve as a kind of "laboratory" for exploring how to generalize quantum mechanics to accommodate nonlinearities.

When N is *assumed* to be linear (and densely defined), Eq. (2) implies that it is a *unitary multiplication operator* for each t. Then N is labeled by a measurable function $\theta(\mathbf{x},t)$, and we have

$$\psi'(\mathbf{x},t) \;=\; (\mathbf{U}_\theta\,\psi)\,(\mathbf{x},t) \;=\; \exp\left[i\theta(\mathbf{x},t)\right]\psi(\mathbf{x},t)\,. \tag{6}$$

Any such $\mathbf{U}_\theta$ commutes with the projection in (3), thus ensuring (5) and respecting the conventional prescription for wave functions subsequent to a positional measurement.

If θ is independent of $\mathbf{x}$ and t, we have just introduced a fixed phase, sometimes called a "gauge transformation of the first kind." This changes neither the Schrödinger equation nor the form of position and momentum operators. A space- and time-dependent, linear $U(1)$-gauge transformation, implemented by (6), is sometimes called a "gauge transformation of the second kind." Such transformations constitute an abelian group $\mathcal{U}_{loc}$ of *local* unitary operators acting on $\mathcal{H}$. The physical equivalence of the two theories, with states ψ and ψ' respectively, is guaranteed by the invariance of the outcomes of sequences of positional observations at all times.

A system with wave functions governed by the (linear) Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\,\psi \tag{7}$$

is transformed by (6) to a physically equivalent system, with wave function ψ' and Schrödinger equation

$$i\hbar\partial_t\psi' = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hbar\nabla\theta\right)^2\psi' + (V - \hbar\partial_t\theta)\,\psi'\,. \tag{8}$$

This observation suggests a *generic way* to construct new systems that are *not* physically equivalent to the original family given by (7)-(8). The scalar term $-\hbar\partial_t\theta$ and the vector term $\hbar\nabla\theta$ are merely special choices. If we take replace $-\hbar\partial_t\theta$ by a *general* scalar field $\hat{\Phi}(\mathbf{x},t)$ and $\hbar\nabla\theta$ by a *general* vector field $\hat{\mathbf{A}}(\mathbf{x},t)$, calling them (abelian) *gauge fields*, we obtain two well-known and well-established structures:

(a) a family $\mathcal{F}_{(\hat{\Phi},\hat{\mathbf{A}})}$ of time-evolution equations labeled by the gauge fields $\hat{\Phi}$ and $\hat{\mathbf{A}}$:

$$i\hbar\partial_t\psi = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - \hat{\mathbf{A}}\right)^2\psi + \left(V + \hat{\Phi}\right)\psi\,; \tag{9}$$

and

(b) an action on this family by the gauge transformations $\mathbf{U}_\theta$, to establish equivalence classes — so that Schrödinger equations with $(\hat{\Phi},\hat{\mathbf{A}})$ and with $(\hat{\Phi}',\hat{\mathbf{A}}') = (\hat{\Phi} - \hbar\partial_t\theta, \hat{\mathbf{A}} + \hbar\nabla\theta)$ describe physically equivalent systems — with the family being closed under the action of gauge transformations.

This generic construction, which we here call *gauge generalization*, is physically relevant because external electromagnetic fields $(\Phi,\mathbf{A})$ interacting with a charged particle provide a realization of $(\hat{\Phi},\hat{\mathbf{A}})$ in nature: in Gaussian units, $\hat{\Phi} = e\Phi$ and $\hat{\mathbf{A}} = (e/c)\mathbf{A}$, where e is the charge of the particle. The gauge-transformed Schrödinger equations are physically equivalent to the original, but those obtained from them by gauge generalization are not. These well-known results provide a model for similar arguments involving the nonlinear transformations N.

In Section II, we demonstrate that two straightforward, physically-motivated conditions precisely specify the group $\mathcal{N}$ of time-dependent, nonlinear gauge transformations introduced in Ref. [1]. These assumptions are: (a) strict locality and (b) a separation condition. We observe that (5) is then ensured.

In Section III, we apply various subgroups of $\mathcal{N}$ to the linear Schrödinger equation (7). This leads to physically equivalent systems satisfying NLSEs, where the coefficients obey certain constraints. Then, in structural analogy to the way (8) motivates (9), we construct new, physically *inequivalent* systems by generalizing the parameters so as to break the constraints.

In Section IV, following this analogy, we consider the parameters as *gauge parameters.* We thus obtain a family of NLSEs through gauge generalization and gauge closure, labeled by the gauge parameters, on which the gauge group acts to establish physical equivalence classes. In this way, we derive naturally — as a unified class — equations containing the terms proposed by Kostin, Bialynicki-Birula and Mycielski, and Doebner and Goldin, with coefficients that are (in general) time-dependent. The subfamily that includes the equations of Guerra and Pusterla turns out to be equivalent to linear quantum mechanics.

of mixed states in nonlinear theories, physical assumptions.

We believe this to offer a fundamentally new perspective, partially elucidating the hidden mathematical and physical structure behind certain nonlinear quantum time-evolutions.

II. CONDITIONS ON NONLINEAR GAUGE TRANSFORMATIONS

A. Locality

We have from Eq. (2) that

$$N[\psi](\mathbf{x},t) = \exp[iG_\psi(\mathbf{x},t)]\psi(\mathbf{x},t)\,, \tag{10}$$

where G_ψ is a real-valued function of $\mathbf{x}$ and t depending on ψ. It is apparent that G_ψ must be further restricted if for instance we hope to ensure (5) for all Borel subsets B of $\mathbf{R}^3$. Suppose the value of $G_\psi(\mathbf{x},t)$ at $\mathbf{x} = \mathbf{x}_1$ depends nontrivially on values of $\psi(\mathbf{x},t)$ for $\mathbf{x} \neq \mathbf{x}_1$ and the evolution equation is local. Then we will be unable to satisfy (4)-(5) in the general case of a region B where $\mathbf{x}_1 \in B$ but $\mathbf{x} \notin B$. Therefore let us assume N to be a *local* transformation, in analogy with the linear gauge transformations $\mathbf{U}_\theta$. This is taken here in the strict sense that the value of $N[\psi]$ at $(\mathbf{x},t)$ should depend only on $\mathbf{x}$, t, and the value of $\psi(\mathbf{x},t)$ — not on any other space or time points, and not on derivatives of ψ. Then we must have

$$\psi'(\mathbf{x},t) = N_F\,[\psi](\mathbf{x},t) = \exp\left[iF(\psi(\mathbf{x},t),\mathbf{x},t)\right]|\psi(\mathbf{x},t)|\,, \tag{11}$$

where F is a real-valued function (defined up to integer multiples of 2π) of the three variables whose values are provided by $\psi(\mathbf{x},t)$, $\mathbf{x}$, and t . The possible dependence of F on the value of $\psi(\mathbf{x},t)$ allows nonlinearity in N_F. With $R(\mathbf{x},t) = |\psi(\mathbf{x},t)|$ and $S(\mathbf{x},t) = \arg\psi(\mathbf{x},t)$, we can consider F to be a function of the real variables $R, S, \mathbf{x}$, and t, relaxing for now the requirement that F take the same value at S and $S + 2\pi n$.

Note that a weaker assumption, in which F is permitted to depend on finitely many derivatives of ψ at $(\mathbf{x},t)$, may still be compatible with (5). We make a stricter assumption here, which limits the resulting time-evolution equations to second order.

B. A separation condition

We consider now systems of n particles described by normalized states in $\mathcal{H}^{(n)} = L^2(\mathbf{R}^{3n}, d^{3n}x)$. For simplicity, take each individual particle to evolve under the same time-evolution operator $T^{(1)}$. We suppose a hierarchy of time-evolutions $T^{(n)}$ of n-particle states, fulfilling the *separation condition.* For linear time evolutions this condition requires that product states $\psi^{(n)} = \psi_1 \otimes \ldots \otimes \psi_n$, $\|\psi_j\| = 1$, $j = 1,\ldots,n$, evolve into product states:

$$T^{(n)}\left[\psi^{(n)}\right] = T^{(1)}[\psi_1] \otimes T^{(1)}[\psi_2] \otimes \ldots \otimes T^{(1)}[\psi_n]\,. \tag{12}$$

It ensures that in the absence of interaction terms, initially uncorrelated subsystems remain uncorrelated, and $T^{(n)}$ is extended (by linearity) from product states to all of $\mathcal{H}^{(n)}$.

It is physically plausible to assume (12) for nonlinear time evolutions $T^{(n)}$ as well [3,17]. Then nonlinear gauge transformations $N_F^{(n)}$ should respect this condition. Here Eq. (11), the states $\psi_j'(\mathbf{x},t) = N_F[\psi_j]$ in (11) are governed by a nonlinear time evolution $T^{(1)'}$, and our separation condition becomes for product states $\psi^{(n)'} = \psi_1' \otimes \ldots \otimes \psi_n'$, $\|\psi_j'\| = 1$, $j = 1, \ldots, n$,

$$T^{(n)'}\left[\psi^{(n)'}\right] = T^{(1)'}[\psi_1'] \otimes \cdots \otimes T^{(1)'}[\psi_n'] \,. \tag{13}$$

We thus want a nonlinear gauge transformation $N_F^{(n)}$ acting on the unit sphere in $\mathcal{H}^{(n)}$, with $\|N_F^{(n)}[\psi]\| = 1$, so that on the product states $\psi^{(n)}$

$$N_F^{(n)}[\psi] = N_F[\psi_1] \otimes \ldots \otimes N_F[\psi_n] \,. \tag{14}$$

Unitary gauge transformations $\mathbf{U}^{(n)}$ in $\mathcal{H}^{(n)}$ may be written as

$$\left(\mathbf{U}\psi^{(n)}\right)(\mathbf{x}_1, \ldots, \mathbf{x}_n, t) = \exp[i\theta_n(\mathbf{x}_1, \ldots, \mathbf{x}_n, t)]\psi(\mathbf{x}_1, \ldots, \mathbf{x}_n, t) \,. \tag{15}$$

On product states, using (6), we want

$$\mathbf{U}^{(n)}\psi^{(n)} = (\mathbf{U}_\theta\psi_1) \otimes (\mathbf{U}_\theta\psi_2) \otimes \ldots \otimes (\mathbf{U}_\theta\psi_n) \,, \tag{16}$$

so that

$$\theta_n(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, t) = \sum_{j=1}^{n} \theta(\mathbf{x}_j, t). \tag{17}$$

And of course, for this case, the operators are linear and can be extended by linearity from product states to the whole Hilbert space. But $N_F^{(n)}$ in Eq. (14) is nonlinear, so we cannot extend it uniquely to $\mathcal{H}^{(n)}$.

The apparently weak condition (14) leads nevertheless to a sharp restriction on F. To see this it is sufficient to discuss the case $n = 2$. With $R_j = |\psi_j|$, $S_j = \arg\psi_j$, $j = 1, 2$, Eqs. (11) and (14) imply that

$$F(R_1, S_1, \mathbf{x}_1, t) + F(R_2, S_2, \mathbf{x}_2, t) \tag{18}$$

depends only on the product $R = R_1R_2$ and the sum $S = S_1 + S_2$. Thus

$$F(R_1, S_1, \mathbf{x}_1, t) + F(R_2, S_2, \mathbf{x}_2, t) = F(R, S, \mathbf{x}_1, t) + F(1, 0, \mathbf{x}_2, t) \,, \tag{19}$$

for all $R, S, \mathbf{x}_1, \mathbf{x}_2, t$, whence $F(R_2, S_2, \mathbf{x}_2, t) - F(1, 0, \mathbf{x}_2, t)$ must be independent of $\mathbf{x}_2$. Setting $F(1, 0, \mathbf{x}, t) = \theta(\mathbf{x}, t)$ and $L(R, S, t) = F(R, S, \mathbf{x}, t) - \theta(\mathbf{x}, t)$, we have the functional equation

$$L(R_1, S_1, t) + L(R_2, S_2, t) = L(R_1R_2, S_1 + S_2, t) \,. \tag{20}$$

The smooth solutions of (20) are given by $L(R, S, t) = \gamma(t) \ln R + \Lambda(t)S$, where γ, Λ are real functions of t. Non-degeneracy of the transformation requires $\Lambda(t) \neq 0$. Finally, we have

$$F(R,S,\mathbf{x},t) = \gamma(t)\ln R + \Lambda(t)S + \theta(\mathbf{x},t)\,. \tag{21}$$

The above argument is similar to the way in which generalized homogeneity of the time-evolution is deduced from the separation property [17].

Note that our separation condition is *weak* in the sense that for nonlinear $T^{(n)}$ and $N_F^{(n)}$ it is only defined on product states; for non-product (entangled) initial states, non-interacting subsystems may yet acquire new correlations. The nonlocal effects in some nonlinear evolution equations can be traced back to this fact [18]. A *strong* version of the separation condition, more adapted to the physical situation and valid for general states, can be formulated along the lines given in Ref. [18].

C. The result

In short, the locality and the separation condition required on n-particle product states boils down the transformations N_F for single particle states to those labeled by two real functions γ and Λ of time, with Λ non-vanishing, and a real function θ of space and time:

$$\left(N_{(\gamma,\Lambda,\theta)}[\psi]\right)(\mathbf{x},t) = |\psi(\mathbf{x},t)|\exp\left[i\left(\gamma(t)\ln|\psi(\mathbf{x},t)| + \Lambda(t)\arg\psi(\mathbf{x},t) + \theta(\mathbf{x},t)\right)\right]\,. \tag{22}$$

The set $N_{(\gamma,\Lambda,\theta)}$ forms a group $\mathcal{G}$, with multiplication law

$$N_{(\gamma',\Lambda',\theta')} \circ N_{(\gamma,\Lambda,\theta)} = N_{(\gamma'+\Lambda'\gamma,\Lambda'\Lambda,\theta'+\Lambda'\theta)}\,. \tag{23}$$

This can be expressed in terms of 3×3 matrices,

$$N_{(\gamma,\Lambda,\theta)} \simeq \begin{pmatrix} 1 & 0 & 0 \\ \theta & \Lambda & 0 \\ \gamma & 0 & \Lambda \end{pmatrix} \tag{24}$$

with entries $\Lambda = \Lambda(t)$, $\gamma = \gamma(t)$, and $\theta = \theta(\mathbf{x},t)$ taken from the corresponding function spaces. We thus have here a group $\mathcal{G}$ of nonlinear gauge transformations, strictly local and separating on n-particle product states, labeled by time-dependent parameters γ and Λ together with a function $\theta(\mathbf{x},t)$. The group is a semi-direct product of the group of gauge transformations of the second kind $\mathcal{U}_{loc} = \{\mathbf{U}_\theta\}$ and the group $\mathcal{N}$, mentioned in the introduction, of 'pure nonlinear' gauge transformations (where $\theta \equiv 0$):

$$\mathcal{G} = \mathcal{N} \otimes_s \mathcal{U}_{loc}\,. \tag{25}$$

$\mathcal{G}$ can be viewed as a *nonlinear generalization* of $\mathcal{U}_{loc}$, the group of 'gauge transformations of the *third kind*' [1].

The transformations $N_{(\gamma,\Lambda,\theta)}$ are not uniquely defined on the Hilbert space. If we restrict the range of Λ to the integers, $\Lambda(t) \in \mathbf{Z}$, then $N_{(\gamma,\Lambda,\theta)}$ is well defined. Then if Λ is a continuous function of time, Λ has to be a constant; $N_{(\gamma,\Lambda,\theta)}$ is invertible with this restriction only for $\Lambda = \pm 1$. $\Lambda = -1$ corresponds to complex conjugation: $N_{(0,-1,0)}\psi = \bar{\psi}$. $N_{(\gamma,1,\theta)}$ is strongly continuous [15], and the set of these transformations is an Abelian subgroup of $\mathcal{G}$,

$$\mathcal{G} \supset \mathcal{G}_0 = \mathcal{N}_0 \otimes \mathcal{U}_{loc}\,. \tag{26}$$

where $\mathcal{N}_0 := \{N_\gamma := N_{(\gamma,\Lambda\equiv 1,\theta\equiv 0)}\}$.

For non-integer Λ, $N_{(\gamma,\Lambda,\theta)}$ may be specified uniquely on certain domains in the Hilbert space, e.g. by imposing continuity of the phase of $N(\psi)$ on a domain of non-vanishing functions ψ. However, such a domain is not needed explicitly for our further considerations.

D. A generalization

Because of the difficulties with the separation condition mentioned above, a more general group structure is also of interest. This can be obtained. without assuming separation, by making a physically-motivated, weaker assumption: an intertwining relation that follows from requiring compatibility with linear gauge transformations.

The group of linear gauge transformations $\mathcal{U}_{loc}$ is commutative, but this need not be the case for the set $\{N_F\} \supset \mathcal{U}_{loc}$. In particular, $\mathbf{U}_\theta$ might not commute with N_F. We explore the condition that N_F be consistent with the usual notion of physical equivalence under gauge transformations of the second kind. That is, the result of applying N_F to a gauge-transformed theory with wave functions $\mathbf{U}_\theta\,\psi$ should be expressible as a transform by $\mathbf{U}_{\theta'}$ of the theory with wave functions $N_F[\psi]$, where, in general, $\theta'(\mathbf{x},t) \neq \theta(\mathbf{x},t)$. Thus we require an *intertwining relation*

$$N_F[\mathbf{U}_\theta\,\psi] = \mathbf{U}_{\theta'} N_F[\psi]\,. \tag{27}$$

Here the function $\theta'(\mathbf{x},t)$ depends on both of the functions F and θ.

Then Eq. (27) implies the functional equation

$$\exp\, i[F(R, S+\theta\,; \mathbf{x},t)] = \exp\, i[\theta'(\mathbf{x},t) + F(R,S;\mathbf{x},t)]\,, \tag{28}$$

valid for each R, S, $\mathbf{x}$, and t. It is straightforward to show that smooth solutions F of (28) take the form $F(R,S;\mathbf{x},t) = k(R,\mathbf{x},t) + \lambda(\mathbf{x},t)S$, where k and λ are real-valued functions of the indicated variables. Non-degeneracy of the transformation requires $\lambda(\mathbf{x},t) \neq 0$ for all $\mathbf{x},t$. Thus N_F is parameterized by k and λ, and given by

$$N_{(k,\lambda)}[\psi](\mathbf{x},t) = \exp\, i[k(|\psi(\mathbf{x},t)|,\mathbf{x},t) + \lambda(\mathbf{x},t)\arg\psi(\mathbf{x},t)]\ |\psi(\mathbf{x},t)|\,. \tag{29}$$

One easily checks that (2), (11), and (27) are fulfilled, with

$$\theta'(\mathbf{x},t) = \lambda(\mathbf{x},t)\theta(\mathbf{x},t)\,. \tag{30}$$

The set $\{N_{(k,\lambda)};\ \lambda(\mathbf{x},t) \neq 0\}$ is a non-commutative, infinite dimensional group $\tilde{\mathcal{G}}$ with multiplication law

$$N_{(k,\lambda)} \circ N_{(k',\lambda')} = N_{(k+k'\lambda,\lambda\lambda')}\,. \tag{31}$$

$N_{(0,1)}$ acts as the identity on ψ, and $N_{(-k/\lambda,\,1/\lambda)}$ is the (formal) inverse of $N_{(k,\lambda)}$. The group law may be expressed as multiplication of 2×2 matrices

$$N_{(k,\lambda)} \simeq \begin{pmatrix} 1 & 0 \\ k & \lambda \end{pmatrix} \tag{32}$$

with entries $k(|\psi|, \mathbf{x}, t)$ and $\lambda(\mathbf{x}, t)$ taken from function spaces. Such matrices span a linear representation $\mathit{Aff}(1)$ of the one-dimensional affine group.

The nonlinear transformations $N_{(\gamma,\Lambda,\theta)}$ are special cases of $N_{(k,\lambda)}$; i.e., the separation condition restricts k and λ to the form

$$k(|\psi|, \mathbf{x}, t) = \gamma(t) \ln |\psi| + \theta(\mathbf{x}, t)\,, \tag{33}$$

$$\lambda(\mathbf{x}, t) = \Lambda(t)\,; \tag{34}$$

and $\mathcal{G}$ is a subgroup of $\tilde{\mathcal{G}}$.

III. NONLINEAR QUANTUM-MECHANICAL EVOLUTION EQUATIONS FROM GAUGE GENERALIZATION

A. Linearizable NLSEs

In accordance with the discussion in Section I, we are now interested in the evolution equation of

$$\psi'(\mathbf{x}, t) = N_{(\gamma,\Lambda,\theta)}[\psi](\mathbf{x}, t)\,, \tag{35}$$

when $\psi(\mathbf{x}, t)$ is a solution of a linear Schrödinger equation

$$i\partial_t \psi = (\nu_1 \Delta + \mu_0 V)\, \psi\,. \tag{36}$$

Let us regard (36) as belonging to a parameterized family $\mathcal{F}_0(\nu_1, \mu_0)$, $\nu_1 \neq 0$, depending on the two real parameters ν_1, μ_0; in Eq. (7), $\nu_1 = -\hbar/2m$ and $\mu_0 = 1/\hbar$.

Due to (27) linear gauge transformations can be treated independently, and we shall here restrict ourselves to the case $\theta \equiv 0$. Applying the group $\mathcal{N}$ to $\mathcal{F}_0$, we obtain a family $\overline{\mathcal{F}_0}$ of NLSEs

$$i\partial_t \psi' = \Big(\nu_1'(t)\Delta + \mu_0'(t)V + F_{DG}^{(0)}[\psi'] + F_{BM}[\psi'] + F_K[\psi']\Big)\, \psi'\,, \tag{37}$$

where

$$F_{DG}^{(0)}[\psi'] = \mu_1'(t) \left(\nabla \left(\mathrm{Im} \left\{ \frac{\nabla \psi'}{\psi'} \right\} \right) + \frac{i}{2} \frac{\Delta |\psi'|^2}{|\psi'|^2} \right) + 2\kappa'(t) \frac{\Delta |\psi'|}{|\psi'|}\,, \tag{38}$$

$$F_{BM}[\psi'] = \alpha_1'(t) \log |\psi'|^2\,, \tag{39}$$

$$F_K[\psi'] = \alpha_2'(t) \arg \psi'\,. \tag{40}$$

The coefficients $\nu_1', \mu_0', \mu_1', \kappa', \alpha_1'$, and α_2' are constrained, and depend on both ν_1, μ_0, and on $\Lambda(t), \gamma(t)$:

$$\begin{gathered} \nu_1'(t) = \frac{1}{\Lambda(t)}\nu_1\,, \quad \mu_0'(t) = \Lambda(t)\mu_0\,, \quad \mu_1'(t) = -\frac{\gamma(t)}{\Lambda(t)}\nu_1\,, \quad \kappa'(t) = \frac{\gamma(t)^2 + \Lambda(t)^2 - 1}{2\Lambda(t)}\nu_1\,, \\ \alpha_1'(t) = \gamma(t)\frac{\dot{\Lambda}(t)}{2\Lambda(t)} - \frac{1}{2}\dot{\gamma}(t)\,, \quad \alpha_2'(t) = -\frac{\dot{\Lambda}(t)}{\Lambda(t)}\,. \end{gathered} \tag{41}$$

This family $\overline{\mathcal{F}_0}$ is closed under $\mathcal{N}$; i.e., it is the *gauge closure* of $\mathcal{F}_0(\nu_1, \mu_0)$ under the action of the group $\mathcal{N}$. It is, up to questions of domain mentioned above, linearizable. It depends on the independent quantities ν_1, μ_0, $\gamma(t)$ and $\Lambda(t)$. One could also write $\overline{\mathcal{F}_0}$ as $\mathcal{F}(\nu_1, \mu_0, \mu_1, \kappa, \alpha_1, \alpha_2)$ labelled by time-dependent coefficients that are constrained.

Note that if Λ and γ are independent of t, the coefficients are time-independent, and $\alpha'_1 = \alpha'_2 = 0$.

If we restrict $\mathcal{N}$ to the subgroup $\mathcal{N}_0$, then starting with $\mathcal{F}_0(\nu_1, \mu_0)$, we obtain a family $\overline{\mathcal{F}_0}^0$ closed under $\mathcal{N}_0$ and contained in $\overline{\mathcal{F}_0}$; here the indexed bar denotes the closure with respect to $\mathcal{N}_0$. The elements in $\overline{\mathcal{F}_0}^0$ are by construction linearizable NLSEs. The parameters are

$$\begin{gathered}\nu'_1 = \nu_1\,, \quad \mu'_0 = \mu_0\,, \quad \mu'_1(t) = \gamma(t)\nu_1\,, \quad \kappa'(t) = \frac{\gamma(t)^2}{2}\nu_1\,, \\ \alpha'_1(t) = -\frac{1}{2}\dot{\gamma}(t)\,, \quad \alpha'_2(t) = 0\,. \end{gathered} \tag{42}$$

Now the term F_K disappears, everything is well-defined, and ν'_1 and μ'_0 are time-independent invariants. Strictly speaking, these NLSEs are *defined* using the continuity and invertibility of $N_{(\gamma,1,0)}$.

For later purposes we mention that $F^{(0)}_{DG}$ decomposes into independent nonlinear real functionals R with the following properties: $R[\psi]$ is Euclidean invariant, complex homogeneous of degree zero and a rational function of $\psi, \bar{\psi}$ with derivatives not higher than second order in the numerator only. There exist five functionals of this type (see [6]):

$$\begin{gathered} R_1[\psi] = \frac{\nabla\cdot\mathbf{J}}{\rho}\,, \quad R_2[\psi] = \frac{\Delta\rho}{\rho}\,, \\ R_3[\psi] = \frac{\mathbf{J}^2}{\rho^2}\,, \quad R_4[\psi] = \frac{\mathbf{J}\cdot\nabla\rho}{\rho^2}\,, \quad R_5[\psi] = \frac{(\nabla\rho)^2}{\rho^2}\,, \end{gathered} \tag{43}$$

where $\rho = \bar{\psi}\psi$ and $\mathbf{J} = (1/2i)\left(\bar{\psi}\nabla\psi - (\nabla\bar{\psi})\psi\right)$ are the probability density and current corresponding to ψ. With this notation $F^{(0)}_{DG}$ in (38) is a complex linear combination:

$$F^{(0)}_{DG}[\psi] = \mu_1(t)\left(R_1[\psi] - R_4[\psi]\right) + i\nu_2(t)R_2[\psi] + \kappa(t)(R_2[\psi] - \frac{1}{2}R_5[\psi])\,, \tag{44}$$

with

$$\nu_2(t) = -\frac{1}{2}\mu_1(t)\,. \tag{45}$$

The term $R_3[\psi]$ will appear in the next section.

B. Generalizing linearizable NLSEs; gauge parameters

The nonlinear gauge transformations $N_{(\gamma,\Lambda)}$ generate special linearizable NLSEs; i.e., nonlinear PDEs with constrained coefficients, physically equivalent to linear Schrödinger equations. Hence the situation is similar to the case of gauge transformations $\mathbf{U}_\theta$ in Section I. It is possible to construct generically through *gauge generalizations* and *gauge closures* a sequence of new families of evolution equations physically *in*equivalent to the linear Schrödinger equation. We obtain the sequence of these families in three steps:

Step 1: We break the constraints (41) in $\overline{\mathcal{F}_0}$ (gauge generalization); i.e., we take the six constrained coefficients $\nu_1', \mu_0', \mu_1', \kappa', \alpha_1', \alpha_2'$ as independent functions of time. Thus we obtain a family $\mathcal{F}_1\,(\nu_1, \mu_0, \mu_1, \kappa, \alpha_1, \alpha_2)$ with six independent parameters. The gauge transformations $\mathcal{N}$ are automorphisms of this family. That is, $\overline{\mathcal{F}_1} = \mathcal{F}_1$; the family is gauge closed. In the notation of Ref. [1], $\kappa = \mu_2 - \frac{1}{2}\nu_1$.

Step 2: We break the constraint (45) for F^0_{DG} in $\mathcal{F}_1$ (gauge generalization),

$$F^{(1)}_{DG}[\psi] = i\nu_2(t) R_2[\psi] + \mu_1(t)(R_1[\psi] - R_4[\psi]) + \kappa(t)(R_2[\psi] - \frac{1}{2}R_5[\psi])\,, \tag{46}$$

and obtain a seven-parameter family $\mathcal{F}_2(\nu_1, \nu_2, \mu_0, \mu_1, \kappa, \alpha_1, \alpha_2)$ of NLSEs (37), with $F^{(1)}_{DG}$ replacing $F^{(0)}_{DG}$.

The action of the group $\mathcal{N}$, however, does not leave this family invariant. The gauge closure $\overline{\mathcal{F}_2}$ of $\mathcal{F}_2$ consists of NLSEs (37) with

$$F^{(2)}_{DG}[\psi] = i\nu_2'(t) R_2[\psi] + \mu_1'(t)(R_1[\psi] - R_4[\psi]) + \kappa'(t) R_2[\psi] + \xi'(t) R_5[\psi] \tag{47}$$

in place of $F^{(1)}_{DG}$. Now there are eight coefficients. The next step is again to break any constraints, but the coefficients are already not constrained. Thus we write $\overline{\mathcal{F}_2}$ as a family $\mathcal{F}_3(\nu_1, \nu_2, \mu_0, \mu_1, \kappa, \xi, \alpha_1, \alpha_2)$ with eight time-dependent parameters, and invariant by construction under $\mathcal{N}$; i.e., $\overline{\mathcal{F}_2} = \mathcal{F}_3 = \overline{\mathcal{F}_3}$. In the notation of ref. [1], $\xi = \mu_5 + \frac{1}{4}\nu_1$. The explicit formula for these coefficients is given by Eq. (50) below, with $\mu_3 = -\nu_1$ and $\xi = -\frac{1}{2}\kappa$.

Step 3: We write $\Delta\psi$ as a complex linear combination of $R_j[\psi]\,\psi$,

$$\Delta\psi = \left(iR_1[\psi] + \frac{1}{2}R_2[\psi] - R_3[\psi] - \frac{1}{4}R_5[\psi] \right)\psi\,, \tag{48}$$

insert into (37) and obtain an additional term $(\mu_3 + \nu_1)R_3[\psi]$ in F_{DG}, and a constraint $\mu_3(t) = -\nu_1(t)$.

We break this constraint, and obtain from $\mathcal{F}_3$ a family $\mathcal{F}_4(\nu_1, \nu_2, \mu_0, \mu_1, \kappa, \mu_3, \xi, \alpha_1, \alpha_2)$ depending on nine time-dependent parameters.

The closure $\overline{\mathcal{F}_4}$ is larger than $\mathcal{F}_4$ and contains all NLSEs (37) with

$$F_{DG}[\psi] = i\nu_2' R_2[\psi] + \mu_1' R_1[\psi] + \kappa' R_2[\psi] + (\mu_3' + \nu_1') R_3[\psi] + \mu_4' R_4[\psi] + \xi R_5[\psi]\,, \tag{49}$$

where the time-dependent coefficients are given by:

$$\begin{gathered}
\nu_1' = \tfrac{\nu_1}{\Lambda}\,, \quad \nu_2' = -\tfrac{\gamma}{2\Lambda}\nu_1 + \nu_2\,, \quad \Lambda\mu_0\,, \quad \mu_1' = -\tfrac{\gamma}{\Lambda}\nu_1 + \mu_1\,, \\
\kappa' = \tfrac{\gamma^2+\Lambda^2-1}{2\Lambda}\nu_1 - \gamma\nu_2 - \tfrac{\gamma}{2}\mu_1 + \Lambda\kappa\,, \quad \mu_3' = \tfrac{1}{\Lambda}\mu_3\,, \quad \mu_4' = \tfrac{\gamma}{\Lambda}\nu_1 - \mu_1 - \tfrac{\gamma}{\Lambda}\mu_3\,, \\
\xi' = \tfrac{1-\gamma^2-\Lambda^2}{4\Lambda}\nu_1 + \tfrac{\gamma}{2}\mu_1 + \tfrac{\gamma^2}{4\Lambda}\mu_3 + \Lambda\xi\,, \\
\alpha_1' = \Lambda\alpha_1 - \tfrac{\gamma}{2}\alpha_2 + \gamma\tfrac{\dot\Lambda}{2\Lambda} - \tfrac{1}{2}\dot\gamma\,, \quad \alpha_2' = \alpha_2 - \tfrac{\dot\Lambda}{\Lambda}\,.
\end{gathered} \tag{50}$$

These coefficients are actually independent, so that $\overline{\mathcal{F}_4}$ is a ten-parameter family. For a more symmetrical notation, we now go over to using $\mu_2 = \kappa + \frac{1}{2}\nu_1$ and $\mu_5 = \xi - \frac{1}{4}\nu_1$, denoting the family by $\mathcal{F}_5(\nu_1, \nu_2, \mu_0, \ldots, \mu_5, \alpha_1, \alpha_2)$:

$$i\partial_t\psi \;=\; i\sum_{j=1}^{2}\nu_j R_j[\psi]\,\psi + \sum_{k=1}^{5}\mu_k R_k[\psi]\,\psi + \mu_0 V\psi + \alpha_1 \log|\psi|^2\psi + \alpha_2(\arg\psi)\psi\,, \tag{51}$$

or in a form which exhibits the linear part separately, with Laplacian Δ,

$$\begin{aligned} i\partial_t\psi \;=\; & (\nu_1\Delta + \mu_0 V)\,\psi + i\nu_2 R_2[\psi]\,\psi \\ & +\mu_1 R_1[\psi]\,\psi + (\mu_2 - \tfrac{1}{2}\nu_1)R_2[\psi]\,\psi + (\mu_3+\nu_1)R_3[\psi]\,\psi \\ & +\mu_4 R_4[\psi]\,\psi + (\mu_5 + \tfrac{1}{4}\nu_1)R_5[\psi]\,\psi \\ & +\alpha_1\log|\psi|^2\psi + \alpha_2(\arg\psi)\psi\,. \end{aligned} \tag{52}$$

$\mathcal{F}_5$ is invariant under the action of the group $\mathcal{N}$; i.e., $\overline{\mathcal{F}_5} = \mathcal{F}_5$.

Starting with the linear family $\mathcal{F}_0$, through iterated gauge generalizations and gauge closures with respect to the pure nonlinear gauge group $\mathcal{N}$, we have thus obtained a sequence

$$\mathcal{F}_0 \subset \overline{\mathcal{F}_0} \subset \mathcal{F}_1 = \overline{\mathcal{F}_1} \subset \mathcal{F}_2 \subset \overline{\mathcal{F}_2} = \mathcal{F}_3 = \overline{\mathcal{F}_3} \subset \mathcal{F}_4 \subset \overline{\mathcal{F}_4} = \mathcal{F}_5 \tag{53}$$

of families of nonlinear Schrödinger equations.

The same procedure can be followed for the restricted gauge group $\mathcal{N}_0$. It turns out that there is an analoguous sequence of families $\mathcal{R}_j$ of NLSEs:

$$\mathcal{F}_0 \equiv \mathcal{R}_0 \subset \overline{\mathcal{R}_0}^0 \subset \mathcal{R}_1 = \overline{\mathcal{R}_1}^0 \subset \mathcal{R}_2 \subset \overline{\mathcal{R}_2}^0 = \mathcal{R}_3 = \overline{\mathcal{R}_3}^0 \subset \mathcal{R}_4 \subset \overline{\mathcal{R}_4}^0 = \mathcal{R}_5\,. \tag{54}$$

The families $\mathcal{R}_j$ are subsets of the $\mathcal{F}_j$:

$$\mathcal{R}_j = \mathcal{F}_j \restriction_{\nu_1(t)=\nu_1,\,\mu_0(t)=\mu_0,\,\mu_3(t)=-\nu_1,\,\alpha_2(t)=0}\,. \tag{55}$$

The only type of term that is not obtained in these families is the term F_K (which is technically not well defined). Note furthermore, that here the parameters of the original linear family $\mathcal{R}_0 \equiv \mathcal{F}_0$ remain invariant, $\nu_1' = \nu_1$ and $\mu_0' = \mu_0$.

IV. DISCUSSION OF THE GAUGE-GENERALIZED NLSE

A. Gauge-invariant parameters, Ehrenfest relations, and Galilei invariance

The group $\mathcal{N}$ transforms the family $\mathcal{F}_5$ into itself. In fact, $N_{(\gamma,\Lambda)}$ acts (for all t) linearly on the eight gauge parameters$\boldsymbol{\nu} = (\nu_1,\ldots,\mu_5)$,

$$\begin{pmatrix} \nu_1' \\ \nu_2' \\ \mu_0' \\ \mu_1' \\ \mu_2' \\ \mu_3' \\ \mu_4' \\ \mu_5' \end{pmatrix} = \begin{pmatrix} \frac{1}{\Lambda} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\gamma}{2\Lambda} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda & 0 & 0 & 0 & 0 & 0 \\ -\frac{\gamma}{\Lambda} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\gamma^2}{2\Lambda} & -\gamma & 0 & -\frac{\gamma}{2} & \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\Lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\gamma}{\Lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\gamma^2}{4\Lambda} & -\frac{\gamma}{2} & \Lambda \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \end{pmatrix}. \tag{56}$$

One can show that the orbits of $\mathcal{N}$, for a fixed time t, are two dimensional on the space $\dot{\mathbf{R}}^8_t := \{\boldsymbol{\nu} \in \mathbf{R}^8_t | \, \nu_1 \neq 0\}$ and foliate $\dot{\mathbf{R}}^8_t$ in two-dimensional leaves. Hence there exist (in general, at least locally; but here in fact globally) six functionally independent parameters $\iota_0, \ldots, \iota_5$ invariant under the action of $\mathcal{N}$ [8,9],

$$\begin{gathered} \iota_0 = \nu_1\mu_0 \,, \quad \iota_1 = \nu_1\mu_2 - \nu_2\mu_1 \,, \quad \iota_2 = \mu_1 - 2\nu_2 \,, \quad \iota_3 = 1 + \mu_3/\nu_1 \,, \\ \iota_4 = \mu_4 - \mu_1\mu_3/\nu_1 \,, \quad \iota_5 = \nu_1(\mu_2 + 2\mu_5) - \nu_2(\mu_1 + 2\mu_4) + 2\nu_2^2\mu_3/\nu_1 \,. \end{gathered} \tag{57}$$

On the remaining two parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, the transformation $N_{(\gamma,\Lambda)}$ acts as an affine transformation,

$$\begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} = \begin{pmatrix} \Lambda & -\frac{\gamma}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\gamma\frac{\dot{\Lambda}}{\Lambda} - \dot{\gamma}) \\ -\frac{\dot{\Lambda}}{\Lambda} \end{pmatrix} . \tag{58}$$

Thus there are two further independent parameters invariant under the action of $\mathcal{N}$ on the *control space* $\dot{\mathbf{R}}^{10}_t$ spanned by $\boldsymbol{\nu}$ and $\boldsymbol{\alpha}$,

$$\iota_6 = \nu_1\alpha_1 - \nu_2\alpha_2 + \nu_2\frac{\dot{\nu}_1}{\nu_1} - \dot{\nu}_2 \,, \quad \iota_7 = \alpha_2 - \frac{\dot{\nu}_1}{\nu_1} \,, \tag{59}$$

generalizing the result in Refs. [8,9] for the family of NLSEs derived in Refs. [5,6]. We call $\boldsymbol{\iota} = (\iota_0, \ldots, \iota_7)$ *gauge-invariant parameters.* They are important for interpreting $\mathcal{F}_5$ and its subfamilies; for details, we refer to Ref. [1], where gauge-invariant parameters have been discussed in a slightly different context.

The subfamilies $\mathcal{F}_j$ and $\mathcal{R}_j$, that are closed under the gauge groups $\mathcal{N}$ and $\mathcal{N}_0$, respectively, can now be characterized in terms of the vanishing of gauge-invariant parameters. Such a characterization is given in Tables 1 and 2, respectively. Note that ν_1 and μ_3 are themselves gauge-invariant parameters of the subfamilies $\mathcal{R}_j$.

TABLES

Table 1

	ι_0	ι_1	ι_2	ι_3	ι_4	ι_5	ι_6	ι_7
$\mathcal{F}_0$	$\times$	$\times$	0	0	0	0	0	0
$\mathcal{F}_1$	$\times$	$\times$	0	0	0	0	$\times$	$\times$
$\mathcal{F}_3$	$\times$	$\times$	$\times$	0	0	$\times$	$\times$	$\times$
$\mathcal{F}_5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Table 2

	ι_1	ι_2	ι_3	ι_4	ι_5	ι_6	ν_1	μ_0
$\mathcal{R}_0$	$\times$	0	0	0	0	0	$\times$	$\times$
$\mathcal{R}_1$	$\times$	0	0	0	0	$\times$	$\times$	$\times$
$\mathcal{R}_3$	$\times$	$\times$	0	0	$\times$	$\times$	$\times$	$\times$
$\mathcal{R}_5$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$

Table 1. Classification of subfamilies of $\mathcal{F}_5$ using gauge-invariants.
Table 2. Classification of subfamilies of $\mathcal{R}_5$ using gauge-invariants.

Some of these families show interesting behaviour. In the family $\mathcal{F}_1$ consider the time dependence of the expectation values $\langle \mathbf{x} \rangle_{\psi(t)} = \int_{\mathbf{R}^3} \mathbf{x} \rho_t(\mathbf{x}) d^3x = \int_{\mathbf{R}^3} \mathbf{x} \overline{\psi(\mathbf{x},t)} \psi(\mathbf{x},t) d^3x$. Then

$$\frac{d}{dt} \langle \mathbf{x} \rangle_{\psi(t)} = -2\nu_1 \langle -i\nabla \rangle_{\psi(t)} \,, \tag{60}$$

$$\frac{d^2}{dt^2} \langle \mathbf{x} \rangle_{\psi(t)} = -2\iota_0 \langle -\mathrm{grad} V \rangle_{\psi(t)} + \iota_7 \frac{d}{dt} \langle \mathbf{x} \rangle_{\psi(t)} \,; \tag{61}$$

i.e., we have the analogue of the first and second Ehrenfest relations for $\mathcal{F}_1$. The center of a non-stationary solution behaves like a classical system under a conservative force and a frictional force proportional to the velocity. For the linearizable subfamily $\overline{\mathcal{F}_0} \subset \mathcal{F}_1$, the frictional term disappears ($\iota_7 = 0$). This is plausible: a linear or nonlinear quantum-mechanical evolution equation and its $\mathcal{N}$-transform describe physically equivalent systems [1].

The first Ehrenfest relation (60) holds for all members of $\mathcal{F}_5$. This shows that the physical systems described by $\mathcal{F}_5$ have something in common. For $\mathcal{F}_2, \ldots, \mathcal{F}_5$ there are additional terms in the second Ehrenfest relation (61), which are connected with the quantum-mechanical diffusion current [5,6].

The free linear SE ($V \equiv 0$) is invariant under the centrally extended Galilei group $G_e(3)$ including time translations. Consider the $G_e(3)$ invariance of $\mathcal{F}_5$ and its subfamilies. $\mathcal{F}_5$ is invariant under $T(t)$, if the gauge-invariant parameters $\boldsymbol{\iota}$ are time independent. If in addition $\iota_3 = \iota_4 = \iota_7 = 0$ the equations are invariant under $G_e(3)$. The generator of time translations is represented via a nonlinear operator H_{nl}, $i\partial_t \psi = H_{nl}[\psi]$, as in Eqs. (51)–(52), while all other generators of $G_e(3)$ are as usual represented linearly. Hence, one has a nonlinear representation of $G_e(3)$ (see also Refs. [8,19,20]).

B. Gauge-generalized NLSE as a unification

Now we are ready to understand the connection between various proposals for nonlinear terms to be added to the linear Schrödinger equation. Such terms have often been chosen in a physically guided, but *ad hoc* way. Some proposed terms have been based directly on fundamental considerations. Our attempt is of the latter type. Its foundation, the physical equivalence of theories and the resulting group of nonlinear gauge transformations (together with gauge generalization and gauge closure) reflects some of the structure of quantum mechanics. Consequently the family $\mathcal{F}_5$ exhibits a common, fundamental basis for some of the proposed NLSEs. Let us consider some of the particular nonlinearities that have been proposed.

1. Logarithmic nonlinearity

Based on the observation that all linear evolution equations for physical quantities are approximations of nonlinear evolutions (except for the Schrödinger equation) Bialynicki-Birula and Mycielski [3] added a (local) nonlinear term $F(|\psi|^2)$. They used the separation property to show that F has to be logarithmic, $F(|\psi|^2) = -b \ln |\psi|^2$. Their NLSE (the BM-family) is

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V - b\ln|\psi|^2\right)\psi\,. \tag{62}$$

This NLSE is contained in $\mathcal{F}_5$ with

$$\nu_1 = \frac{\hbar}{2m}, \quad \mu_2 = \frac{\hbar}{4m}, \quad \mu_3 = -\frac{\hbar}{2m}, \quad \mu_5 = -\frac{\hbar}{8m}, \quad \mu_0 = \frac{1}{\hbar}, \quad \alpha_1 = -\frac{b}{\hbar}, \tag{63}$$

and the other coefficients vanishing. Note that in order to obtain this logarithmic term in our gauge generalization, we had to allow for a time-dependent group parameter $\gamma = \gamma(t)$.

2. Nonlinearity proportional to the phase

One of many examples of a heuristic implementation of dissipation in quantum mechanics is the approach by Kostin [2]. Starting with a frictional term proportional to the expectation of the momentum operator in the (second) Ehrenfest relation, Kostin motivated adding a nonlinear term proportional to the phase of ψ to the linear Schrödinger equation; i.e. (with $f \in \mathbf{R}$),

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V + \frac{\hbar f}{m}\arg\psi\right)\psi\,. \tag{64}$$

Kostin's NLSE (the K-family) is contained in $\mathcal{F}_5$ with

$$\nu_1 = \frac{\hbar}{2m}, \quad \mu_2 = \frac{\hbar}{4m}, \quad \mu_3 = -\frac{\hbar}{2m}, \quad \mu_3 = -\frac{\hbar}{8m}, \quad \mu_0 = \frac{1}{\hbar}, \quad \alpha_2 = \frac{f}{m}, \tag{65}$$

and the other coefficients vanishing. To obtain this term in our approach, we had to assume that $\Lambda = \Lambda(t)$ can be a function of time. Obviously, $\arg\psi$ is not well defined; this is reflected in the problem of gauge transformations with $\Lambda \neq \pm 1$, discussed in Section II C.

3. Nonlinearity from diffeomorphism group representations

The approach of Doebner and Goldin [5,6] is motivated by fundamental considerations. The generic kinematical symmetry algebra $S(\mathbf{R}^3)$ on $\mathbf{R}^3$ is a semidirect sum of the Lie algebra of real smooth functions $f \in C^\infty(\mathbf{R}^3)$, and the Lie algebra of vector fields $X \in \mathrm{Vect}(\mathbf{R}^3)$, or equivalently a local current algebra on $\mathbf{R}^3$ [21–23]. $\mathrm{Vect}(\mathbf{R}^3)$ is the Lie algebra of a subgroup of the group of diffeomorphisms of $\mathbf{R}^3$ (diffeomorphisms trivial at infinity). The functions $f \in C^\infty(\mathbf{R}^3)$ can be interpreted physically as classical position observables and the vector fields $X \in \mathrm{Vect}(\mathbf{R}^3)$ as classical kinematical momenta. Then a quantization map $\mathcal{Q}$ represents the kinematical algebra $S(\mathbf{R}^3)$ by self-adjoint operators in the single particle Hilbert space $\mathcal{H}^{(1)}$. Under physically motivated assumptions, all such representations $\mathcal{Q}$ can be classified up to unitary equivalence by a real parameter D with the dimensionality of a diffusion coefficient [length^2/time]. The presence of such a family of inequivalent representations reflects the richness of $\mathrm{Vect}(\mathbf{R}^3)$. The method can be generalized to any smooth manifold [24].

To obtain some information about the evolution equation of ψ, local probability conservation (for pure states) is assumed [5], or a generalized first Ehrenfest relation is postulated

[25,26]. Then the time-dependent probability density and current are related through an equation of Fokker-Planck type,

$$\partial_t \rho = -\frac{\hbar}{m} \nabla \cdot \mathbf{J} + D \Delta \rho \,. \tag{66}$$

This restricts the evolution equation of ψ to the form

$$i\hbar \partial_t \psi = \left(-\frac{\hbar^2}{2m} \Delta + V \right) \psi + i \, \frac{\hbar D}{2} \frac{\Delta \rho}{\rho} \, \psi + R[\psi] \, \psi \,, \tag{67}$$

where $R[\psi]$ is an arbitrary real-valued (nonlinear) operator. The form of the pure imaginary functional, $\Delta\rho/\rho$, is enforced. If $R[\psi]$ is assumed to be of a similar form, i.e., if it is (i) complex homogeneous of degree zero, (ii) a rational function with derivatives of no more than second order occuring only in the numerator, and (iii) invariant under the 3-dimensional Euclidean group $E(3)$, then a five parameter family of NLSEs (the DG-family) is obtained:

$$R[\psi] = \hbar D' \sum_{j=1}^{5} c_j R_j[\psi] \,, \tag{68}$$

with the R_j as in Eq. (43). Obviously this is a special case of $\mathcal{F}_5$, where $\alpha_1 = \alpha_2 = 0$ and all gauge parameters are time-independent:

$$\begin{aligned} &\nu_1 = -\frac{\hbar}{2m} \,, \quad \nu_2 = \frac{\hbar D}{2} \,, \quad \mu_0 = \frac{1}{\hbar} \,, \quad \mu_1 = \hbar D' c_1 \,, \quad \mu_2 = \hbar D' c_2 - \frac{\hbar}{4m} \,, \\ &\mu_3 = \hbar D' c_3 + \frac{\hbar}{2m} \,, \quad \mu_4 = \hbar D' c_4 \,, \quad \mu_5 = \hbar D' c_5 + \frac{\hbar}{8m} \,, \quad \alpha_1 = \alpha_2 = 0 \,. \end{aligned} \tag{69}$$

The equation proposed by Guerra and Pusterla in connection with de Broglie's double solution theory [4] is contained in this family, with $D = 0$, $c_1 = c_3 = c_4 = 0$, $c_5 = -\frac{1}{2} c_2$.

V. SUMMARY

To summarize, we have taken a small step toward a nonlinear quantum theory which could be physically relevant, by discussing nonlinear evolution equations derived from fundamental considerations.

Under the assumption that all measurements are positional measurements performed at different times, we derived a group of nonlinear gauge transformations $\mathcal{G}$, including the usual linear ones. Applying these transformations to a linear Schrödinger equation, we obtained nonlinear ones, and after gauge generalization and gauge closure we reached a family $\mathcal{F}_5$ of nonlinear Schrödinger equations. Certain subfamilies of $\mathcal{F}_5$ were motivated originally by different physical ideas and different mathematical structures. Thus $\mathcal{F}_5$ is a *unification* of these NLSEs: the BM-family, the K-family, and the DG-family. It is surprising, and also satisfying, when different structures and lines of reasoning yield the same or compatible results. This is an indication that these structures have a common origin. If there is some deeper reason for this, beyond the gauge generalization process described here, we have not yet unveiled it.

Moreover, our discussion may show how to circumvent some formal arguments against nonlinear quantum theory put forth by Gisin and others [27–29]; in connection with nonlocal effects, we refer especially to [18]. We have not touched on other problems of nonlinear quantum theory, such as the concept of mixed states (see Ref. [16]), or discussed the physical interpretation of a (necessarily non-selfadjoint) nonlinear Hamiltonian.

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